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Salt Water Intrusion in Aquifers: Development and Application of a Vertically Integrated Two-Dimensional Finite-Element Model

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ABSTRACT

Salt water intrusion along an irregular coastline and under offshore islands often is a result of ground-water development. On a regional scale vertical components of flow and diffusion in the mixing zone between the fresh water and the salt water are frequently negligible. In this case, a sharp interface between fresh water and salt water is a valid approximation. In this paper the simultaneous solution of the vertically integrated linked equations for flow in the fresh water and salt water is performed. The model GWCH2O is presented which is a vertically integrated finite-element mathematical model developed to simulate the problem of salt water intrusion in coastal aquifers in two dimensions.

Applications and verification of the model GWCH2O include: 1) steady-state position of the salt water - freshwater interface in an unconfined aquifer, 2) the development of a fresh-water lens over a salt water body, and 3) the salt water upconing beneath a coastal pumping well.

INTRODUCTION

Salt water intrusion in a coastal aquifer is the classical example of a density-dependant problem in ground-water hydrology. The economical importance of this problem has invited continuous research since the late nineteenth century. The analyses of salt water intrusion problems can be separated into two broad categories according to the method used to handle the transition zone between the freshwater and the salt water. The freshwater and salt water can be considered as a single density dependent fluid or as two miscible fluids separated by a transition zone. Frequently the thickness of the transition zone is small compared to the aquifer thickness. In this case, a sharp interface can be assumed between the freshwater and salt water. This assumption is particularly useful for large-scale areal problems.

In the vertically integrated approach, a separate equation is written in three dimensions for the freshwater and the salt water. These equations are integrated over the vertical dimension resulting in a set of linked two-dimensional equations. This approach was first presented by Shamir and Dagan (1971). They considered a vertical cross-section and used vertical integration to develop one-dimensional equations, which were solved using the finite-difference method. For this one-dimensional case they were able to track the toe of the interface and regenerate the grid for each time step. Bonnet and Sauty (1975) extended the work of Shamir and Dagan to two dimensions. Pinder and Page (1976) used the same equations given by Bonnet and Sauty using the Galerkin finite-element method of solution. Wilson and Sa da Costa (1982) developed a one-dimensional finite-element model with indirect toe tracking using a fixed grid. Their procedure used the Gaussian quadrature points for toe tracking, a nonlinear variation of fluid saturation for those elements containing a moving boundary and an imaginary thickness of the locally absent fluid.

Sakr (1992) extended the work of Wilson and Sa da Costa to two-dimensions. In this model he assumed that salt water and freshwater exist everywhere with infinitesimal thickness even when the fluid is locally absent. This assumption enables the model to simulate toe problems, as well as lens problems, without any loss of generality. Sakr used the Galerkin finite-element method with linear triangular elements and linear shape functions.

In this paper a two-dimensional finite-element model, GWCH2O, is presented which is capable of solving the freshwater salt water problem by the vertically integrated sharp interface approach. This method has the advantage that it avoids the use of a moving grid or grid regeneration. The model is verified by comparison between the analytical solution and the model results for three problems: 1) the steady-state position of the freshwater salt water interface in an unconfined aquifer, 2) development of a freshwater lens over salt water body, and 3) the salt water upconing beneath a coastal pumping well.

MATHEMATICAL DEVELOPMENT OF GOVERNING EQUATIONS

The vertically integrated linked flow equations for freshwater and salt water are presented in detail in Sakr (1992). In this paper, a new approach will be introduced using the mass balance approach.

The Dupuit assumption of horizontal flow in both fresh and salt water domains permits the use of a material balance control volume [see Figure (1)], that extends from the bottom of the aquifer to the top of the salt water domain (ie the freshwater - salt water interface). Another control volume extends from the interface to the water surface for the freshwater domain. For

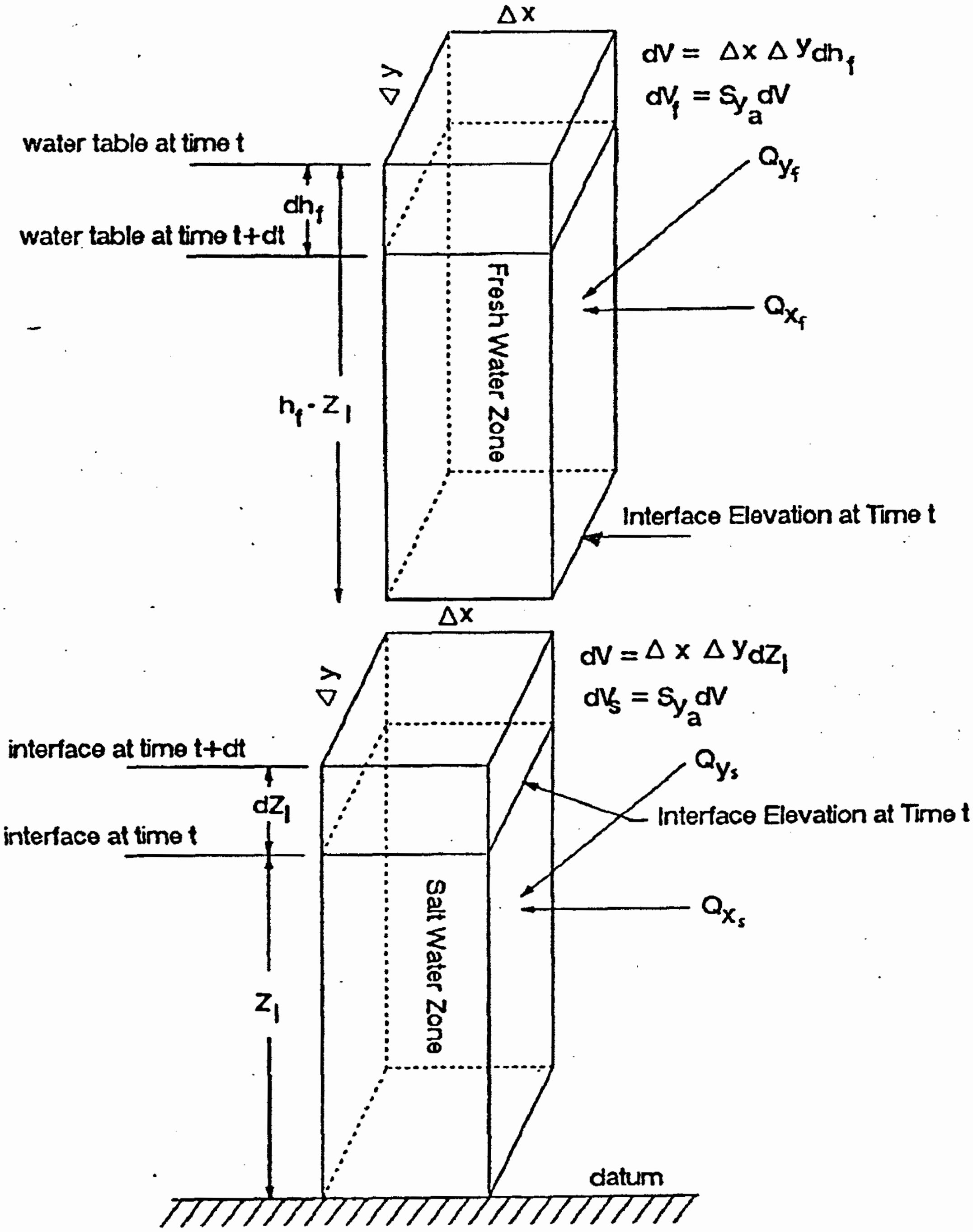

FIGURE 1 - Control volume in an unconfined coastal aquifer

$$\textit{the net outflow rate} = \frac{\partial Q_{x_s}}{\partial x} \Delta x + \frac{\partial Q_{y_s}}{\partial y} \Delta y \; ;$$

$$Q_{x_s} = -K_{xx_s} B_s \frac{\partial h_s}{\partial x} \Delta y \; , \quad Q_{y_s} = -K_{yy_s} B_s \frac{\partial h_s}{\partial y} \Delta x \tag{1}$$

$$\frac{\textit{net outflow rate}}{\Delta x \Delta y} = -\frac{\partial}{\partial x}\left((K_{xx})_s B_s \frac{\partial h_s}{\partial x}\right) - \frac{\partial}{\partial y}\left((K_{yy})_s B_s \frac{\partial h_s}{\partial y}\right) . \tag{2}$$

The net rate of outflow must equal the negative of the time rate of change of the stored water volume. The change in the storage can be expressed as

$$-\left(\frac{\partial V_w}{\partial t}\right)_s = S_s \frac{\partial h_s}{\partial t} \Delta x \Delta y + \Phi \frac{\partial Z_I}{\partial t} \Delta x \Delta y.$$

Combining equations 2 and 3 yields the governing partial differential equation for the salt water (equation 4). Using a similar approach the governing partial differential equation for the freshwater is obtained (equation 5). Notice, that any increase in storage of salt water results in a decrease in freshwater storage.

$$\frac{\partial}{\partial x}\left((K_{xx})_s B_s \frac{\partial h_s}{\partial x}\right) + \frac{\partial}{\partial y}\left((K_{yy})_s B_s \frac{\partial h_s}{\partial y}\right) = S_s \frac{\partial h_s}{\partial t} + \Phi \frac{\partial Z_I}{\partial t}$$

$$\frac{\partial}{\partial x}\left((K_{xx})_f B_f \frac{\partial h_f}{\partial x}\right) + \frac{\partial}{\partial y}\left((K_{yy})_f B_f \frac{\partial h_f}{\partial y}\right) = S_f \frac{\partial h_f}{\partial t} - \Phi \frac{\partial Z_I}{\partial t} + \alpha\Phi \frac{\partial h_f}{\partial t} .$$

The above two equations are coupled through the interface elevation term. Using the Hubbert theory, the fluid pressure must be continuous across the salt water and freshwater interface. This requirement leads to the relationship

$$Z_I = (1 + \delta)\, h_s - \delta\, h_f$$

$$\frac{dZ}{dt} = (1 + \delta) \frac{\partial h_s}{\partial t} - \delta \frac{\partial h_f}{\partial t} .$$

Substituting equation 6 into equations 4 and 5 yields Sakr (1992). For the salt water:

$$\frac{\partial}{\partial \chi}\left(B_s(K_{xx})_s\frac{\partial h_s}{\partial x}\right)+\frac{\partial}{\partial y}\left(B_s(K_{yy})_s\frac{\partial h_s}{\partial y}\right)=((S)_sB_s+\phi(1+\delta))\frac{\partial h_s}{\partial t}-\phi\delta\frac{\partial h_f}{\partial t}+Q_s \ .$$

and for the freshwater:

$$\frac{\partial}{\partial x}\,(B_f\,(K_{xx})_f\,\frac{\partial\, h_f}{\partial\, x}\,)\,+\,\frac{\partial}{\partial\, y}\,(B_f\,(K_{yy})_f\,\frac{\partial\, h_f}{\partial\, y}\,=\,\alpha\;\Phi\;\frac{\partial h_f}{\partial t}\,+\,\alpha R\,+$$

$$(\,(S)_f\,B_f\,+\,\phi\;\delta\;)\;\frac{\partial\, h_f}{\partial t}\,-\,(\delta+1)\frac{\partial h_s}{\partial t}\,+\,Q_f\ \ .$$

Finite-Element Solution

Equations (7) and (8) are solved using the Galerkin finite-element method. Because the procedure of the Galerkin formulation is now standard and can be found in several references, there is no need to include it in this presentation. Thus, only the final set of equations that need to be solved is given in matrix form as

$$\textit{for salt water}\quad [A_s]\{\,h_s\,\}\,+\,[B_s]\left(\frac{dh_s}{dt}\right)\,+\,[C_s]\left(\frac{dh_f}{dt}\right)\,+\,\{\,E_s\,\}\,=\,0$$

$$\textit{for fresh water}\quad [A_f]\{\,h_f\,\}\,+\,[B_f]\left(\frac{dh_f}{dt}\right)\,+\,[C_f]\left(\frac{dh_s}{dt}\right)\,+\,\{\,E_f\,\}\,=\,0.$$

[A], [B] and [C] are n*n dimensional matrices and {E, h, dh/dt} are n dimensional vectors with the appropriate subscript, $(_f)$, for freshwater and $(_s)$, for salt water. The elements of matrices [A], [B], [C] and vector {E} are defined by:

$$[A_{ij}]=\iint_D\left[(T_{xx})\frac{\partial\phi_i(x,y)}{\partial x}\frac{\partial\phi_j(x,y)}{\partial x}+(T_{yy})\frac{\partial\phi_i(x,y)}{\partial y}\frac{\partial\phi_j(x,y)}{\partial y}\right]dA$$

$$[B_{ij}]=\iint_D[\epsilon\phi_i(x,y)\phi_j(x,y)]dA,\quad [C_{ij}]=-[\Gamma\phi_i(x,y)\phi_j(x,y)]dA$$

$$\{ E_{ij} \} = \iint_D \left[\phi_i(x,y) \left(\sum_{K=1}^{m} (\delta(x-x_K)\ \delta(y-y_K)\ Q_{tK}) \right) \right] dA \tag{13}$$

where $\epsilon_s = S_s B_s + \Phi(\delta+1)$, $\Gamma_s = \Phi\delta$, $(T_{xx})_s = B_s(K_{xx})_s$, $(T_{yy})_s = B_s(K_{yy})_s$

$\epsilon_f = S_f B_f + \Phi\delta + \alpha\Phi$, $\Gamma_f = \Phi(1+\delta)$, $(T_{xx})_f = B_f(K_{xx})_f$, $(T_{yy})_f = B_f(K_{yy})_f$

Solution Scheme

To solve equations (9) and (10) it is necessary to evaluate the integral which appears in the coefficient matrices. In the model GWCH2O, linear functions that are defined over triangular subspaces are used. This allows the integrations to be carried out in closed form which is computationally very efficient. The time derivatives in equations 9 and 10 are evaluated by finite-difference method as:

$$\frac{dh}{dt} = \frac{h_{n+1} - h_n}{\Delta t} \,. \tag{14}$$

VERIFICATION AND APPLICATION

PROBLEM 1 -- Steady-State Interface in an Unconfined Aquifer

The analytical solution for this problem is presented in Sakr (1992) as:

$$Z_I(X) = h_s - \sqrt{\frac{2\delta^2 Q_o}{K(\delta+1)} X + SF \,.\, \frac{\delta-1}{\delta+1} (\frac{\delta Q_o}{K})^2} .$$

$$h_f = h_s + \sqrt{\frac{2Q_o}{K(\delta+1)} X + SF \,.\, (\frac{\delta-1}{\delta+1})^{\frac{1}{2}} (\frac{Q_o}{K})^2} .$$

where X is measured inland from the coastline. The parameter values and the boundary conditions are shown in Figure (2).

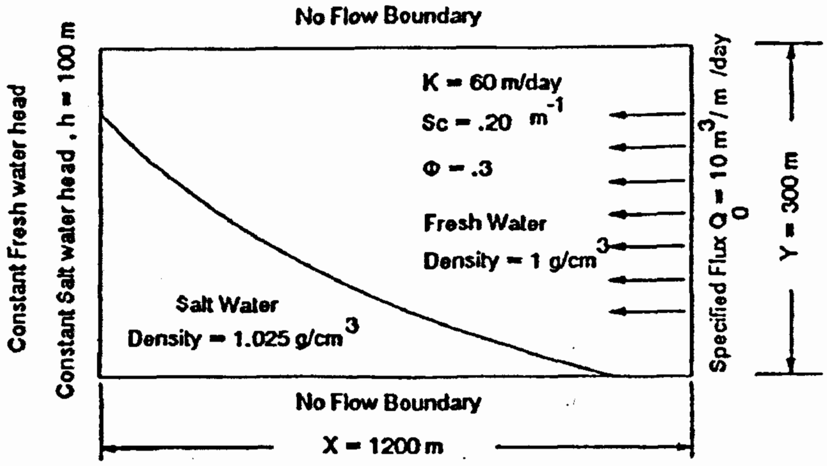

FIGURE 2- Parameter values used to simulate a steady-state interface in an unconfined coastal aquifer.

Figure (3) shows the results from the numerical model GWCH2O with the analytical solution. As can be readily seen the numerical results match the analytical solution very closely.

PROBLEM 2 – Development of a Lens of Freshwater Over Salt Water Body

The model GWCH2O was developed to solve the governing equations of two miscible fluids of differing densities and separated by a sharp interface (this also assumes the existence of a toe or tip). To judge performance of the model to solve other types of problems, the transient development of a lens for a lighter fluid (freshwater) floating on top of a heavier one (salt water) is chosen. For this example, a coastal aquifer with finite width L and a uniform recharge from an infinitely long strip parallel to the coast was simulated with the parameters shown in Figure (4). The initial condition for the interface and the phreatic surface was specified as a horizontal plane with the elevation equal to the average sea level. This problem was simulated twice. The first case assumes that the salt water body is static to check the validity

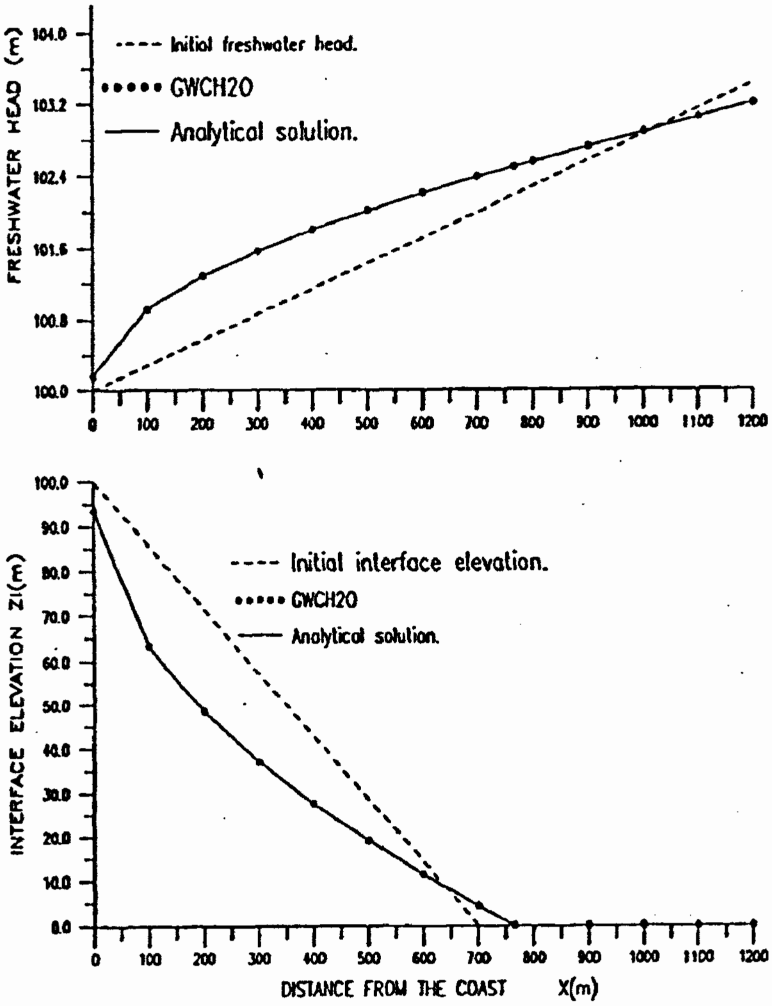

FIGURE 3- Comparison between GWCH2O and the analytical solution for the steady-state interface in unconfined aquifer.

and the limitation of the Ghyben-Herzberg approximation when transient cases are considered. The second case considers the dynamics of the salt water domain. Figure (5) shows a comparison between the two simulations at different time intervals. It is clear that the results at the early times (10 and 50 days) and the results for the steady-state solution are nearly identical. However, there are differences for the other cases at the later times (100, 300, and 500 days). The reason is that at early times of a dynamic system there is not enough elapsed time for the applied stress to cause a significant change in the salt water head. Also, for the time later than 100 days for this particular problem there is a change in the salt water head which causes salt water to be trapped and which causes a change in the interface elevation compared to the static case. So, use of the Ghyben-Herzberg formula should be avoided in simulating these problems unless the steady-state solution or the position of the interface at the very early or very late times

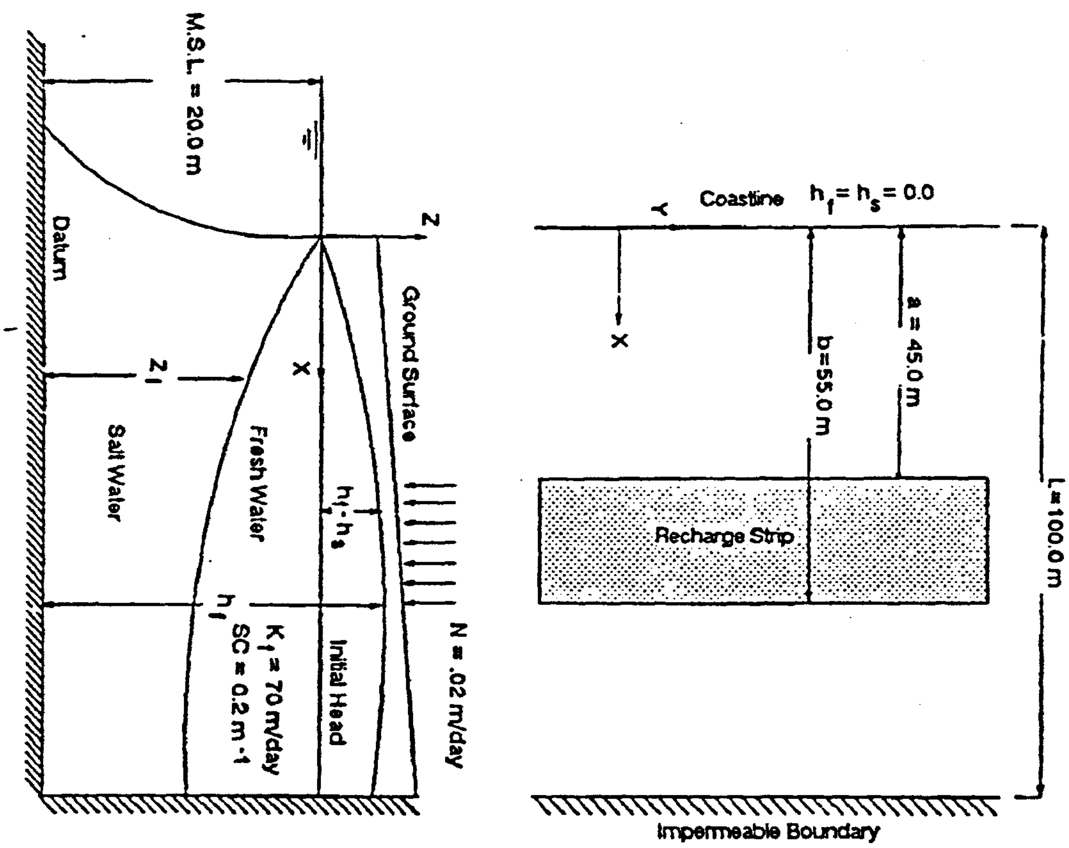

FIGURE 4-Schematic representation of the development of a freshwater lens over salt water

are desired.

PROBLEM 3 -- Salt Water Upconing Beneath a Coastal Pumping Well

In coastal aquifers pumping wells are used for water supply. The steady-state position of the interface and the freshwater head distributions are determined for different pumping rates to set a pumping strategy. Assuming that the well does not penetrate the salt water zone the following parameter values with the boundary conditions and the finite-element mesh shown in Figure (6) are considered. The mesh is finer near the well in order to capture the sharp gradients of piezometric head. The grid covers only half of the domain space due to the symmetry of the

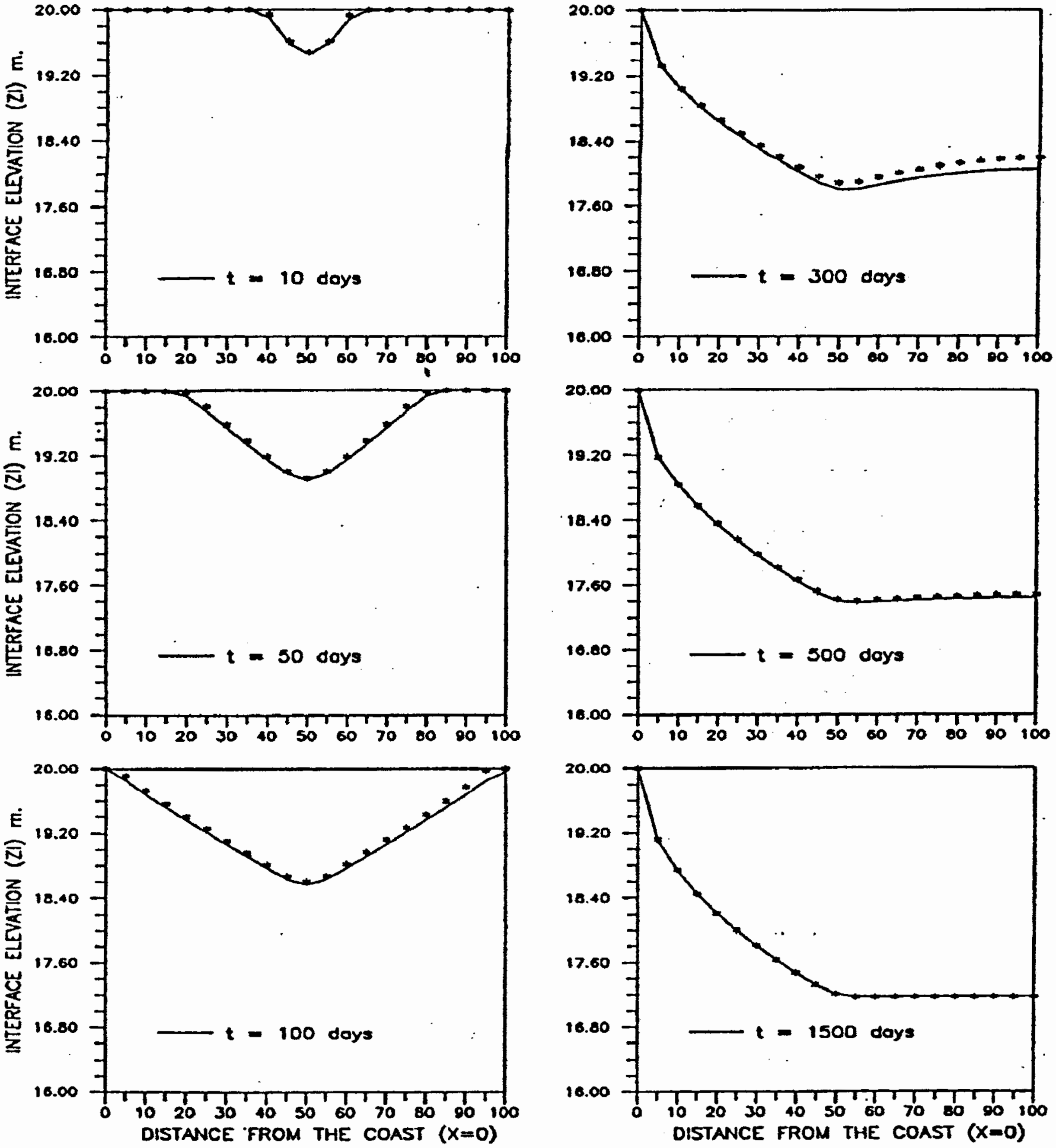

FIGURE 5 -Development of a freshwater lens over salt water
——— GWCH2O with Ghyben-Herzberg assumption.
**** GWCH2O without Ghyben-Herzberg assumption.

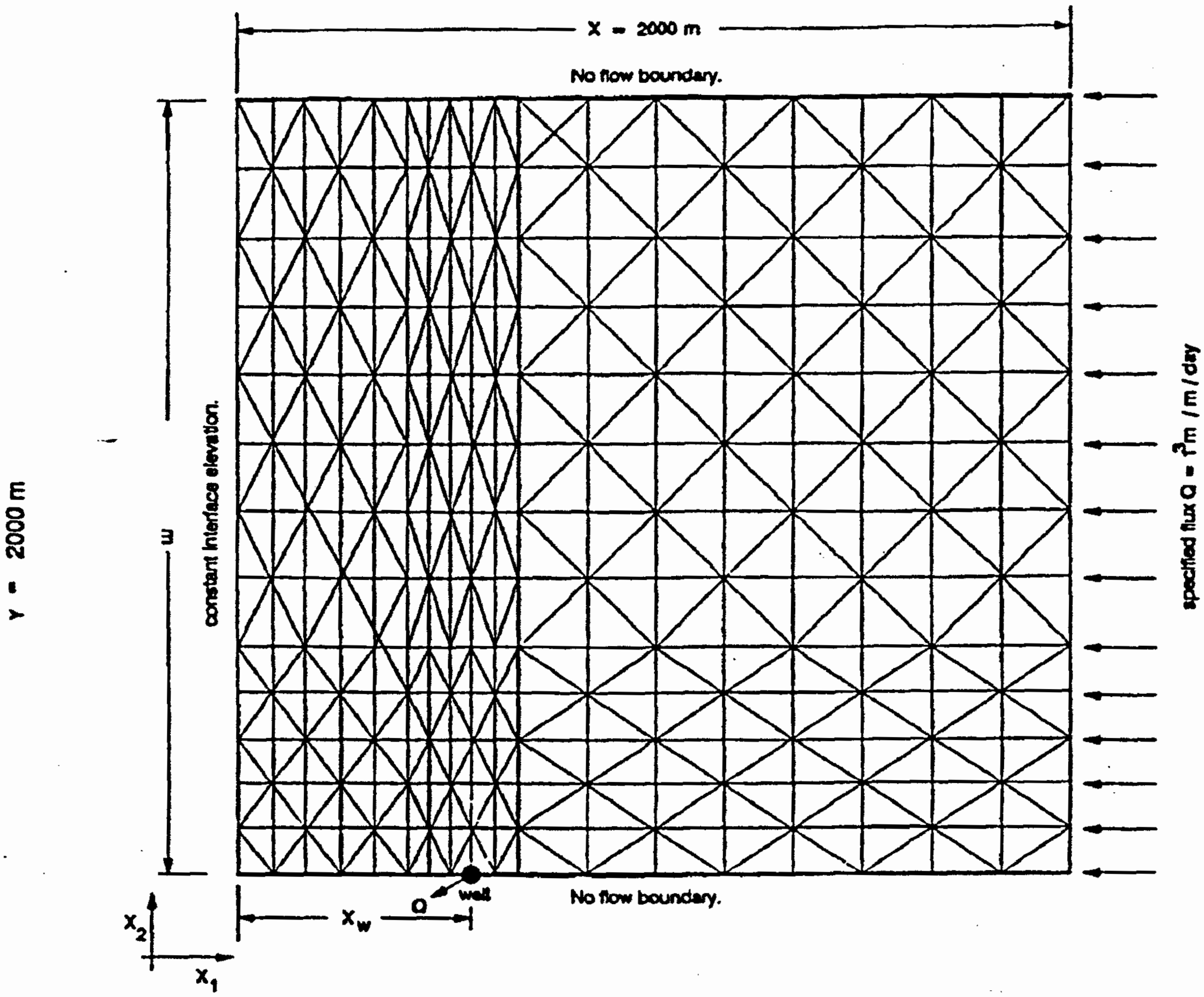

FIGURE 6-Finite-element mesh and aquifer parameters used to simulate coastal pumping well problem.

The first simulation performed assumed the well discharge $Q_w = 0$ in order to determine the initial position of the interface. The results of this simulation are the initial conditions for the following pumping simulations. At times $t > 0$ the pumping well is pumped at different pumping rates of $Q_w = 400, 300, 200\ m^3/day$. Figure (7) shows the steady-state interface and the head distribution at $x_2 = 0.0$ m for the different pumping rates. It is obvious that there is a rise in the interface beneath the pumping well because of the drawdown of the piezometric surface. Substantial vertical components of flow occur in the vicinity of the well which will introduce errors into the prediction of the interface from the Ghyben-Herzberg equation. However, the conclusion that small drawdowns caused by the pumping well will result in large rises of the interface remain valid. Therefore, extraction of freshwater from a coastal aquifer must be

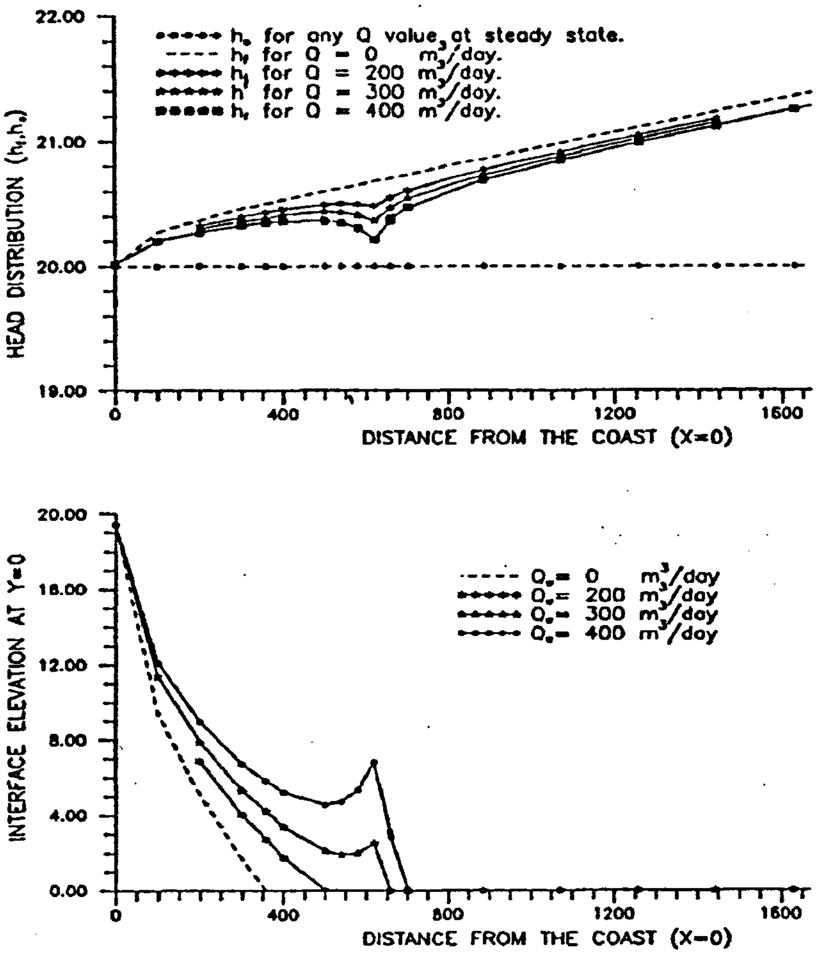

FIGURE 7- Steady-state interface and head distributions for different pumping rates in an unconfined coastal aquifer.

accomplished by creating a very small drawdown if salt water is to be prevented from upconing into the pumping wells. To obtain a more accurate prediction for a safe discharge this problem should be studied in three-dimensions.

CONCLUSIONS

The model GWCH2O has been shown to be a versatile, accurate and efficient method of studying the problem of salt water intrusion in coastal aquifers. The model accurately represents aquifer situations in which a salt water toe occurs and no toe tracking algorithm is involved.

For the freshwater lens problem, the model GWCH2O confirmed that the Ghyben-Herzberg approximation is not usually valid for transient problems.

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NOTATION

s	salt water domain
f	freshwater domain
h	$Z + P/\gamma$ is the hydraulic head [L]
P	fluid pressure [M L^{-1} T^{-2}]
γ	specific weight [M L^{-2} T^{-2}]
Z_B	elevation of the aquifer base [L]
Z_I	elevation of the interface [L]
Z_T	elevation of the top of the fresh water region [L]
B_s	$Z_I - Z_B$ [L]
B_f	$Z_T - Z_I$ [L]
Φ	aquifer Porosity [Dimensionless]
S	specific storage [L^{-1}]
K_{xx}	horizontal components of the hydraulic conductivity in the x direction [LT^{-1}]
K_{yy}	horizontal components of the hydraulic conductivity in the y direction [LT^{-1}]
T_{xx}	horizontal components of the aquifer transmissivity in the x direction [L^2T^{-1}]
T_{yy}	horizontal components of the aquifer transmissivity in the y direction [L^2T^{-1}]
t	time [T]
Q	source/sink term (negative for sink) [T^{-1}]
α	1 for unconfined aquifer and α=0 for confined aquifer
R	recharge rate to the water table [LT^{-1}]
x,y,z	cartesian coordinates in the principal direction of the hydraulic conductivity [L]
δ	$\frac{\gamma_f}{(\gamma_s - \gamma_f)}$